($\vec{v}$ is the "orthogonal projection of $\vec{b}$")

(Solns)

Math 2D Quiz 2 Afternoon - January 21, 2016

Please put name and ID on *both* sides for grading and redistribution!

Show all of your work. *There is a question on the back side.

1 pt each

Based on 12.3.45,46

1. (a) Give the definition of a vector projection of $\mathbf{b}$ onto $\mathbf{a}$. In other words, express $\text{proj}_{\mathbf{a}}\mathbf{b}$.
(b) Let $\mathbf{a} = <0, 1, 4>$, $\mathbf{b} = <0, 2, 3>$. Compute the vector $\mathbf{v} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$.
(c) Compute $\mathbf{b} \times \mathbf{a}$.
(d) What is $\mathbf{v} \times \mathbf{a}$? You must either directly compute this, or explain your answer.
(e) Determine if $\mathbf{v}$ is perpendicular to $\mathbf{a}$. Justify your answer.

a) $\text{proj}_{\vec{a}} \vec{b} = \left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$.

b) Here, $\vec{a} \cdot \vec{b} = 2 + 12 = 14$; $|\vec{a}|^2 = \sqrt{1+16}^{\,2} = 17$

so, $\text{proj}_{\vec{a}} \vec{b} = \dfrac{14}{17} < 0, 1, 4 > = < 0, \dfrac{14}{17}, \dfrac{56}{17} >$

Thus, $\vec{v} = < 0, 2, 3 > - < 0, \dfrac{14}{17}, \dfrac{56}{17} > = < 0, \dfrac{20}{17}, -\dfrac{5}{17} >$

c) $\vec{b} \times \vec{a} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \hat{i} - \det \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix} \hat{j} + \det \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \hat{k}$

$= \boxed{5\hat{i}}$

Or, $\vec{b} \times \vec{a} = (b_2 a_3 - b_3 a_2)\hat{i} + (b_3 \underline{a_1} - \underline{b_1} a_3)\hat{j} + (\underline{b_1} a_2 - b_2 \underline{a_1})\hat{k}$

☆ Note: $b_1 = 0$, $a_1 = 0$ so $\vec{b} \times \vec{a} = (8-3)\hat{i} + 0\hat{j} + 0\hat{k} = \boxed{5\hat{i}}$

d) $\boxed{\vec{v} \times \vec{a} = \vec{b} \times \vec{a} = 5\hat{i}}$ because $\vec{v} = \vec{b} - (\text{scalar})\vec{a}$. Since $\vec{a} \times \vec{a} = \vec{0}$ always,

$\vec{v} \times \vec{a} = \vec{b} \times \vec{a} - \left(\dfrac{14}{17} \right) \underbrace{\vec{a} \times \vec{a}}_{\text{zero}} = \vec{b} \times \vec{a} = \underline{5\hat{i}}$

e) $\vec{v} \cdot \vec{a} = < 0, \dfrac{20}{17}, -\dfrac{5}{17} > \cdot < 0, 1, 4 > = \dfrac{20}{17} - \dfrac{5 \cdot 4}{17} = 0$

since $\vec{v} \cdot \vec{a} = 0 \iff \boxed{\vec{v} \perp \vec{a}}$ ✓ (yes it's perpendicular)

Based on 12.1.35

2. (a) Graph the region in $\mathbb{R}^3$ described by

2 pts

$$4 \le x^2 + y^2 + z^2 \le 9.$$

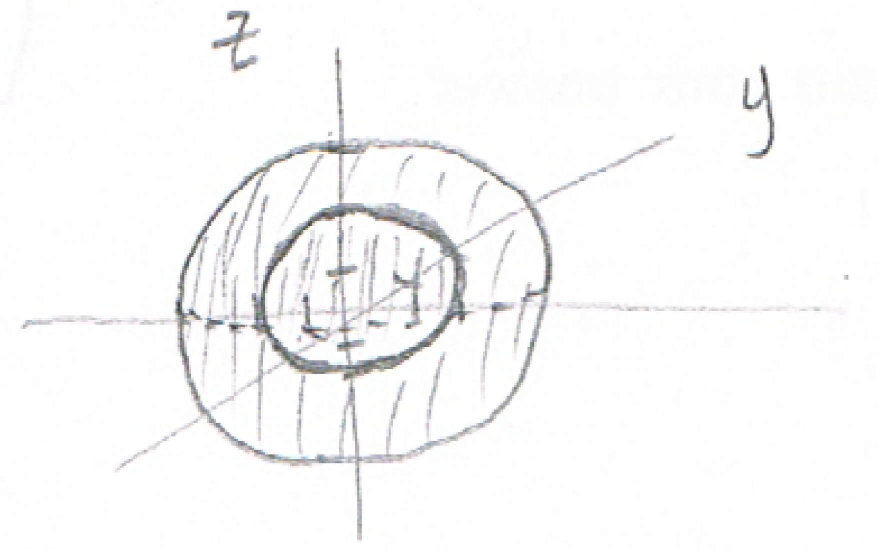
Based on 12.4.43

3 pts

Describe the region that you graphed - be brief.

(b) If $\mathbf{a} \cdot \mathbf{b} = -3\sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = < 1, 2, 2 >$, determine the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

a)



→ spherical shell, whose outer radius is 3 and inner cavity radius is 2. (The shell is solid)

b) $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta_{ab}$ and $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta_{ab}$

So, $-3\sqrt{3} = |\vec{a}||\vec{b}| \cos\theta_{ab}$

$3 = |\vec{a}||\vec{b}| \sin\theta_{ab}$

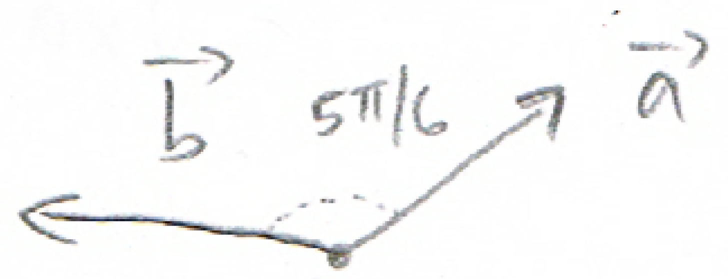
Here, $|\vec{a} \times \vec{b}| = \sqrt{1+4+4}$
$= \sqrt{9} = 3$

Dividing, $\tan\theta_{ab} = \dfrac{3}{-3\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$.

$$\boxed{\therefore \theta_{ab} = -\frac{\pi}{6}, \frac{5\pi}{6}}$$

* Since $\vec{a} \cdot \vec{b}$ is negative, θ_{ab} must be greater than 90° $(\frac{\pi}{2})$ in magnitude, so $\boxed{\theta_{ab} = \frac{5\pi}{6}}$ here.

EX.



↳ Negative Dot Product Implies the vectors are facing "away" from each other.